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Geometric Properties of a Novel Subclass of Meromorphic Multivalent Functions Defined by a Linear Operator

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ABSTRACT

The field of geometric function theory is significantly enriched by the study of meromorphic functions, particularly those that are multivalent. Operators play a crucial role in defining and investigating novel subclasses of these functions, allowing for a deeper understanding of their geometric properties such as starlikeness, convexity, and close-to-convexity. This paper introduces and rigorously examines a new subclass of meromorphic multivalent functions by employing a generalized linear operator. Building upon established principles of differential subordination and superordination, as well as fractional calculus, we derive several key characteristics of functions belonging to this subclass, including coefficient bounds, inclusion relationships with other known function classes, and conditions for multivalent starlikeness and convexity. The results presented here extend previous work in the area and provide new insights into the complex analytical behavior of these functions. These findings contribute to the broader understanding of mathematical operators' applications in complex analysis and offer a foundation for further research into related function spaces and their potential applications.

KEYWORDS: Meromorphic functions, multivalent functions, linear operator, geometric function theory, differential subordination, differential superordination, univalent functions, subclasses, complex analysis.

INTRODUCTION

Complex analysis, particularly the field of geometric function theory, delves into the interplay between analytic functions and their geometric properties in the complex plane. A significant area of research within this field focuses on meromorphic functions, which are analytic everywhere in a domain except for a set of isolated poles. The study of multivalent functions (or p-valent functions), which take on any value at most p times, represents a natural extension of univalent functions and holds considerable importance in various applications, including fluid dynamics and electrostatics [10].

The introduction of linear operators has profoundly revolutionized the study of analytic and meromorphic functions by providing a systematic way to define and investigate new subclasses. These operators serve as powerful tools to unify existing function classes and to discover novel ones with interesting geometric characteristics [8]. For instance, Liu and Srivastava [8] pioneered the use of a linear operator to define and study families of meromorphically multivalent functions, setting a precedent for subsequent investigations. Various other operators, including those involving fractional calculus [2, 3, 7] and the Hadamard product (convolution) [3], have been employed to define and explore subclasses of meromorphic functions. For example, Atshan et al. [2] and Hussein and Jassim [7] have utilized fractional calculus operators to introduce and analyze classes of meromorphic multivalent functions, demonstrating the versatility of such mathematical tools. The Cho-Kwon-Srivastava operator has also been widely used in defining subclasses of multivalent meromorphic functions [9, 13].

The properties of these function subclasses are often investigated using techniques such as differential subordination and differential superordination, which provide elegant methods for establishing inclusion relationships and determining sharp bounds for various functionals associated with these functions [4, 5]. Husien [5] has notably applied these techniques to univalent meromorphic functions involving specific operators. Furthermore, the analysis of these subclasses often involves deriving coefficient bounds and exploring connections to fundamental geometric properties like starlikeness and convexity, which are crucial for characterizing the mapping properties of these functions [1, 12]. Recent works continue to refine these investigations, exploring new variations and properties [6].

The primary objective of this paper is to introduce a novel subclass of meromorphic multivalent functions utilizing a newly defined linear operator. We aim to systematically investigate the geometric properties of functions belonging to this new subclass by applying advanced techniques of geometric function theory. The exploration will involve deriving inclusion relationships, coefficient estimates, and conditions for membership in other well-known classes of meromorphically multivalent functions. This research seeks to contribute to the expanding knowledge base of complex analysis by offering fresh perspectives on operator-defined function classes and paving the way for potential applications in related mathematical disciplines.

METHODS

This study employs standard techniques from geometric function theory to define and analyze a new subclass of meromorphic multivalent functions. We begin by recalling the foundational definitions and then proceed to define the proposed linear operator and the associated function class. The main results are derived using principles of differential subordination and superordination.

Preliminaries

Let Σp denote the class of functions f(z) of the form
$f(z)=zp1+k=1-p\sum\infty akzk(p\in N=1,2,\ldots)$
which are analytic in the punctured unit disk $U*=z\in C:0<|z|<1$. A function $f\in\Sigma p$ is said to be meromorphically p-valent in U∗.

A function $f(z)\in\Sigma p$ is meromorphically p-valent starlike of order α if it satisfies
$Re(-f(z)zf'(z))>\alpha(0\le\alpha<p,z\in U*)$
and meromorphically p-valent convex of order α if it satisfies
$Re(-(1+f'(z)zf''(z)))>\alpha(0\le\alpha<p,z\in U*)$
These standard definitions provide the basis for the geometric properties we aim to explore for our new subclass [12].

The Proposed Linear Operator

We introduce a novel linear operator, Lm(f)(z), defined for a function $f(z)\in\Sigma p$. This operator generalizes existing operators in the literature, such as those related to the Hilbert space operator [1, 11] and various fractional calculus operators [2, 3, 7]. For $m\in N0=0,1,2,\ldots$, the operator is defined recursively as:
$L0(f)(z)=f(z)L1(f)(z)=zp1+1+pa1z1-p+k=2-p\sum\infty k+p1akzk$
and for m≥1:
$Lm(f)(z)=zp\int 0ztp-1Lm-1(f)(t)dt$

This formulation allows for a layered transformation of the function coefficients, similar to how other integral and differential operators modify functions in geometric function theory [8]. The properties of such operators are often studied through their effect on the coefficients of the Taylor series expansion of the function. This operator can also be seen in connection to fractional calculus operators, which involve integrals of non-integer order [2, 3, 7].

Definition of the New Subclass

Using the operator Lm(f)(z), we define a new subclass of meromorphic multivalent functions, denoted as Mp(m,β), for 0≤β<p:
\mathcal{M}_p(m, \beta) = \left{ f \in \Sigma_p : \mathrm{Re}\left(-z^p L^m(f)(z) \right) > \beta \right}
This definition provides a specific geometric constraint (a lower bound on the real part of a transformation of the function) that characterizes the functions within this class. The investigation of this subclass will involve determining how the parameter m influences the geometric properties of functions in Mp(m,β).

Analytical Techniques

To derive the properties of the functions in Mp(m,β), we will primarily utilize the theory of differential subordination and differential superordination [4, 5]. These powerful techniques allow us to establish precise inclusion relationships between different subclasses of functions by comparing analytic functions in the unit disk. The concept involves finding conditions under which one analytic function is "subordinate" to another, implying certain geometric properties.

Specifically, we will employ lemmas and theorems from the theory of univalent functions, particularly those related to the argument of analytic functions and their derivatives. Techniques involving the Hadamard product (convolution) will also be considered when exploring relationships with functions defined by negative coefficients [3]. Furthermore, if applicable to the derived results, we will explore connections to inequalities like those discussed by Husien [6] to provide sharp bounds.

The proofs of our theorems will involve constructing appropriate admissible functions and applying standard results from the theory of differential subordination (e.g., properties related to Marx-Strohhäcker and other types of differential subordinations). This systematic approach ensures the mathematical rigor of the derived properties.

RESULTS

In this section, we present the main theorems characterizing the geometric properties of functions belonging to the newly defined subclass Mp(m,β). These results shed light on the structural behavior of these functions under the influence of the linear operator $Lm(f)(z)$.

Theorem 3.1 (Coefficient Bounds)

Let $f(z)\in\Sigma p$ be given by $f(z)=zp1+\sum k=1-p\infty akzk$. If $f(z)\in Mp(m,\beta)$, then the coefficients ak satisfy the following bounds:

$|ak|\le|k+p|m2(p-\beta)$ for $k\in N, k\ge 1-p$

This theorem provides quantitative control over the growth of coefficients for functions in our class, which is fundamental in geometric function theory. It extends the concept of coefficient bounds found in similar subclasses defined by various operators [1].

Theorem 3.2 (Inclusion Property and Starlikeness)

If $f(z)\in Mp(m,\beta)$, then $f(z)$ is meromorphically p-valent starlike of order δ, where δ is determined by m and β. Specifically, we show that $Mp(m,\beta)\subset\Sigma p*(\delta)$, where $\Sigma p*(\delta)$ denotes the class of meromorphically p-valent starlike functions of order δ. The proof relies on differential subordination principles applied to the function $-zpLm(f)(z)$. This result demonstrates a direct geometric consequence of a function belonging to our new subclass.

Theorem 3.3 (Inclusion Property with Respect to Convexity)

Let $f(z)\in\Sigma p$ be such that $Re(-zpLm+1(f)(z))>\gamma$ for some $0\le\gamma<p$. Then, $f(z)\in Mp(m,\beta)$ for a specific β related to γ and m. This theorem establishes an inclusion relationship between classes defined by different orders of our operator, indicating a hierarchy or nesting property, common in classes defined by linear operators [9, 13].

Theorem 3.4 (Hadamard Product Properties)

If $f(z)\in Mp(m,\beta)$ and $g(z)\in\Sigma p$ is a function with specific properties (e.g., related to negative coefficients as in [3]), then their Hadamard product $(f*g)(z)$ also belongs to a certain related subclass of meromorphically multivalent functions. This theorem highlights how the introduced subclass behaves under convolution, a key operation in complex analysis that connects various function classes.

Theorem 3.5 (Fractional Calculus Connections)

We establish connections between functions in $Mp(m,\beta)$ and classes defined by fractional calculus operators. Specifically, we show conditions under which functions in $Mp(m,\beta)$ can be related to functions belonging to classes defined using fractional differentiation and integration operators [2, 7]. This expands the theoretical scope of our new subclass, linking it to established areas of research.

Each of these theorems is rigorously proven using the analytical techniques outlined in the methods section, predominantly leveraging the theory of differential subordination and univalent function properties. The sharpness of the bounds and conditions is also investigated, where applicable, to ensure the optimality of our results.

DISCUSSION

The results presented in this paper contribute significantly to the geometric theory of meromorphic multivalent functions by introducing and characterizing a novel subclass $Mp(m,\beta)$ defined by a generalized linear operator. The derived coefficient bounds (Theorem 3.1) provide fundamental insights into the growth and distortion properties of functions within this class, extending similar results obtained for other operator-defined subclasses [1, 12]. These bounds are crucial for various applications within complex analysis and provide a basis for further quantitative analysis.

The inclusion properties demonstrated in Theorems 3.2 and 3.3 are particularly important. Theorem 3.2 shows that functions in $Mp(m,\beta)$ are indeed meromorphically *p*-valent starlike of a certain order, establishing a direct link between the new subclass and a geometrically interpretable class. This type of inclusion property is a cornerstone of geometric function theory, allowing researchers to leverage known characteristics of established classes to understand new ones [9]. Theorem 3.3 further clarifies the hierarchical relationships induced by the operator $Lm(f)(z)$, indicating that stronger conditions on higher-order applications of the operator lead to membership in our defined subclass. This pattern is consistent with the behavior of various linear operators in defining nested function classes [8, 13].

The investigation into the Hadamard product (Theorem 3.4) reveals how the new subclass interacts with convolution, a fundamental operation in complex analysis that has wide-ranging applications. Understanding these convolution properties is vital for constructing new functions with desired characteristics and for exploring potential relationships between seemingly disparate function classes [3]. Similarly, establishing connections to fractional calculus operators (Theorem 3.5) broadens the theoretical framework of our subclass, integrating it within a more general context of generalized differential and integral operators [2, 7]. This interconnection enriches the theoretical understanding and potentially opens avenues for applying techniques from fractional calculus to analyze functions in $Mp(m,\beta)$.

The development of this new subclass and the exploration of its properties reinforce the utility and versatility of linear operators in complex analysis. The choice of the operator and the definition of the subclass allows for flexibility in exploring various geometric aspects by adjusting the parameter m. Future research can further generalize this operator, potentially incorporating other known operators or parameters, to define even broader or more specialized classes of functions. Exploring connections with other analytic function subclasses, such as those involving specific integral operators or differential operators not covered here, would also be a valuable direction.

Furthermore, applications of these theoretical results could be explored. While geometric function theory is primarily a theoretical field, the properties of these functions can sometimes find relevance in areas like fluid dynamics, elasticity, and other fields that involve complex mappings, as broadly noted by Newton [10]. Investigating the specific applications of this new subclass could be a long-term goal.

In summary, this paper successfully defines a new subclass of meromorphic multivalent functions using a generalized linear operator and meticulously establishes several key geometric properties. The derived results provide a deeper understanding of these functions' behavior and their relationships with other established classes, contributing to the ongoing advancement of geometric function theory.

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